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God's Omniscience: A Formal Analysis in Normal and Non-normal Epistemic Logics[*]

1. Introduction

One of the major problems which the logician willing to model knowledge and belief has to face is that of avoiding, or at least alleviating, the problem of omniscience. The efforts are usually focused on creating models for agents (either human or artificial) with bounded rationality and finite cognitive capabilities: such agents thus do not possess complete information about how the world is. Logical omniscience is often seen, therefore, as a problem to be solved and the solutions proposed so far are numerous.[1]

Nevertheless, if the issue to be addressed is that of defining divine omniscience, such a perspective should be reversed in order to push the concept of knowledge to its most extreme possibilities. The Christian thesis that God is omniscient is well-established, as the fact that God possesses the most perfect knowledge of all things follows

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[1] R. Fagin, J.Y. Halpern, Y. Moses, and M.Y. Vardi, *Reasoning about Knowledge*, MIT Press, Cambridge–London 1995; J.J. Meyer, *Modal Epistemic and Doxastic Logic*, [in:] D. Gabbay and F. Guenthner (eds), *Handbook of Philosophical Logic*, 2nd edition, Kluwer, Dordrecht 2001.

from His infinite perfection.[2] We shall not investigate the theological grounds of this idea. Rather, we will simply try to clarify the basic meaning of omniscience by pushing modal epistemic logic to its limits. From this perspective, an interesting question is that, even in the case of God's knowledge, omniscience should have conceptual limits, otherwise the risk is to trivialize and make the notion of knowledge void. Perhaps, one may wonder whether this conclusion is problematic: after all, the transcendent nature of God exceeds our rational and conceptual resources. Since in God all things that ought to be are in fact the case (which trivializes the concept of normativity), we can likewise accept that divine knowledge about some A is equivalent to saying that A is true.

This having been stated, one may think that omniscience is quite an easy property to get and hence to formalise. However, this is only partially true. If, on the one hand, it is quite easy to define *logical* omniscience in terms of knowledge of the logical truths,[3] on the other, it turns out to be rather difficult to formally capture the insight of *factual* omniscience, which has to do with propositions having a different status.[4] As we shall see, as soon as we try to capture the intuition behind divine knowledge, the whole scenario becomes foggy and slippery and all of a sudden even the strongest modal logics, those which are usually supposed to naturally define omniscient agents, turn out to be rather useless.

A first informal definition may be that the agent that has complete or maximal knowledge is omniscient. What this informal definition means precisely is the research issue that we address here. Indeed, the aim of this paper is that of providing a formal account of such property. The Classical Propositional Calculus (henceforth CPC) being our foundation, we shall proceed by analyzing those modal schemata and rules which, on our view, are better suited to capture the intuitions

[2] L. Zagzebski, *Omniscience*, [in:] C. Meister and P. Copans (eds), *The Routledge Companion to Philosophy of Religion*, Routledge, London–New York 2007.

[3] R. Fagin et al., *Reasoning..., op. cit.*

[4] R. Girle, *Modal Logic and Philosophy*, Acumen, Teddington 2000.

behind the property of omniscience. At first glance, an omniscient agent may be regarded as a being possessing complete knowledge about both those facts which are necessarily true and those which are just contingently so. Hence, it seems that there are two main types of omniscience. We shall call the first type *logical* and the second *factual*. Intuitively, this last account of divine omniscience looks the strongest candidate.

The layout of the paper is as follows: Section 2 offers a brief outline of the modal logic language, the axiom schemata and inference rules, and the resulting modal systems that we discuss regarding the notion of divine omniscience. Section 3 focuses on a first family of epistemic logics expressing a very weak degree of omniscience: all logics are characterized in the most general semantic setting for modal logics, which, however, can hardly be conceptually re-framed in terms of well-known standard semantics for epistemic logics. A different path, which adopts a direct generalization of such standard semantics is depicted in Section 4. Section 5 considers stronger well-known epistemic logics, where we can have just a single accessibility relation (standard Kripke semantics) or, in the extreme case, where knowledge no longer requires the use of any accessibility relation. Section 6 goes beyond the threshold of logical omniscience and defines the idea of factual omniscience: this allows us to approximate God's perfect knowledge, but the cost to pay is to trivialize epistemic logic. A short summary ends the paper.

2. Preliminaries

As usual, we define the language of CPC as a containing countable set of propositional letters $Prop := \{p, q, \ldots\}$, propositional constants $\top$, $\bot$ (top and bottom), round brackets, boolean operations $\neg, \wedge, \vee, \rightarrow, \equiv$ (negation, conjunction, disjunction, implication, double implication) and one modal unary operator $\Box_i$ for each agent *i* operating in the system (the index *i* being dropped whenever only one agent is supposed to be present). An expression like $\Box_i A$ means that agent i knows/believes that A is true. Well-formed formulae (henceforth wffs) are defined as follows (where p is a propositional letter and A, B are meta-variables for well-formed formulae):

$$p \mid \top \mid \bot \mid \neg A \mid A \wedge B \mid A \vee B \mid A \rightarrow B \mid A \equiv B \mid \Box_i A$$

Let us recall some well-known inference rules and schemata we shall use for any $\Box_i$:[5]

Inference Rules

RE:= $\vdash A \equiv B \Rightarrow \vdash \Box_i A \equiv \Box_i B$
RM:= $\vdash A \rightarrow B \Rightarrow \vdash \Box_i A \rightarrow \Box_i B$
RN:= $\vdash A \Rightarrow \vdash \Box_i A$
RR:= $\vdash A \wedge B \rightarrow C \Rightarrow \vdash \Box_i A \wedge \Box_i B \rightarrow \Box_i C$
RK:= $\vdash A_1 \wedge \dots A_n \rightarrow B \Rightarrow \vdash \Box_i A_1 \wedge \dots \wedge \Box_i A_n \rightarrow \Box_i B \; n \geq 0$

Axiom Schemata

efg:=	$A \wedge \neg A \rightarrow B$	**D:=**	$\Box_i A \rightarrow \neg \Box_i \neg A$
M:=	$\Box_i (A \wedge B) \rightarrow \Box_i A \wedge \Box_i B$	**T:=**	$\Box_i A \rightarrow A$
C:=	$(\Box_i A \wedge \Box_i B) \rightarrow \Box_i (A \wedge B)$	**4:=**	$\Box_i A \rightarrow \Box_i \Box_i A$
K:=	$\Box_i (A \rightarrow B) \rightarrow (\Box_i A \rightarrow \Box_i B)$	**B:=**	$A \rightarrow \Box_i \Diamond_i A$
N:=	$\Box_i \top$	**5:=**	$\Diamond_i A \rightarrow \Box_i \Diamond_i A$
Con:=	$\neg \Box_i \bot$	**DEX:=**	$\Box_i A \wedge \Box_i \neg A \rightarrow \Box_i B$

There are different systems of propositional modal logics built to model various situations. In Figure 1, we consider some simple and well-known multi-modal systems which may be seen as a *base* for more complex ones.

[5] We use B.F. Chellas' terminology and assume as usual that $\Diamond_i =_{def} \neg \Box_i \neg$ (cf. B.F. Chellas, *Modal Logic*, Cambridge University Press, Cambridge 1980).

	Rules	Axioms
E, classical	**RE**	
M, monotonic	**RM**	E ⊕ **M**
MN, N-monotonic	**RM** ⊕ **RN**	E ⊕ **M** ⊕ **N**
R, regular	**RR**	E ⊕ **M** ⊕ **C**
K, normal	**RK**	E ⊕ **K** ⊕ **N**
KD, normal		K ⊕ **D**
T, normal		K ⊕ **T**
B, normal		E ⊕ **K** ⊕ **N** ⊕ **B**
S4, normal		K ⊕ **T** ⊕ **4**
S5, normal		K ⊕ **T** ⊕ **4** ⊕ **5**

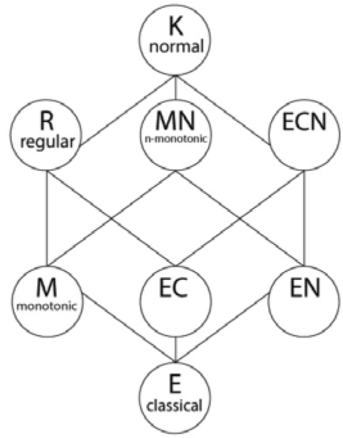

Figure 1: Modal systems

3. The starting point: minimal epistemic logics

Any system of epistemic logic, if based on the standard modal-logic paradigm,[6] should assume some minimal formal properties. In particular, it is well-known that any modal logic should at least be closed under logical equivalence.[7] This will be our starting point to formally analyse the notion of omniscience.

3.1. Principle of co-extensionality: the minimal epistemic logic E

When dealing with CPC, one standard and well-known option is the adoption of a Fregean approach to semantics. In a given state of affairs, propositions are taken to be different names of the only two semantical objects populating the universe: Truth and Falsehood. A tautology is a proposition which is true only in virtue of its logical form: the truth values of its components do not influence the truth of the whole in the slightest. The set of tautologies can, hence be

[6] J.J. Meyer, *Modal Epistemic..., op. cit.*
[7] B.F. Chellas, *Modal Logic, op. cit.*

described as the class of all the *true* names of Truth: those propositions whose truth is certain and unchangeable. A most famous result in formal logic states that all theorems of CPC are tautologies and *vice versa*.

Two propositions that *always* share the same extension can be regarded as logically equivalent and, in a logical sense, identical. This can be expressed symbolically as $A \equiv B$: whichever truth value A is given, it would be identical to B's and *vice versa*.

A basic requirement the knowledge base of a divine (omniscient) being should meet is the principle that for any *known* sentence A, all its equivalents are also known. We are not yet including anything in God's knowledge base. What we are stating is merely that if something is known, then all those facts which 'look' different but are actually the same (logically, extensionally) must also be known. This is a well-known modal principle and it can be compared to Leibniz's Law, here applied to propositional logic. What this principle states is that if two propositions are logically equivalent, they are epistemically interchangeable. Following B.F. Chellas'[8] terminology, this rule shall be henceforth referred to as **RE**:

$$\mathbf{RE} := A \equiv B \ / \ \Box A \equiv \Box B.$$

It can be added to CPC to generate the minimal system of Classical Propositional Modal Logic E. Built on the foundation of propositional logic, the system E becomes our first step towards a logical definition of the concept of divine omniscience.

When knowledge and belief are modelled in epistemic logics like E, which are much weaker than K (see above, Figure 1), then the epistemic logics can have a peculiar semantic reading, which is suitable to provide a fine-grained interpretation of logical omniscience. Modal logics weaker than K – which are called generically *non-normal*, in contrast with any *normal* logics that are stated to be as strong as,

[8] *Ibidem.*

or stronger than, K[9] – can be interpreted in semantic structures that consist of:

- a set of possible worlds or states
- a set of accessibility relations connecting pairs of worlds.

The introduction of a plurality of worlds connected via a given accessibility relation *R* stems in epistemic logics from the need to represent agents' relative ignorance (i.e. partial knowledge) about the world. Given a state *w*, all the *R*-associated worlds *t* are seen as epistemic alternatives to *w* itself[10]: when we have such a relation *R* which connects a world *w* with all alternatives where *A* is true, then we can say that $\Box A$ is true in a world *w* – and $\Box A$ is meant to say that an agent knows/believes that *A* is true. The plurality of worlds captures the notion of partial knowledge as follows. Suppose an agent *i* lives in Paris and does not know whether today it is raining in London (*p*:= 'It is raining in London'). If *i* does not have access to any reliable source of information, he simply ignores all facts about the weather in London, hence he has at least two epistemic alternatives: for *i* from the perspective of Paris, (1) *p* is true, (2) *p* is false. However, as soon as the agent gains access to new pieces of information concerning the meteorological situation of London, the number of alternatives that he considers possible drops. If, for instance, he reads that it is currently raining in London, the epistemic alternatives he considers are only those which reflect the real situation, i.e. only those in which the proposition *p* is true. His knowledge base would then change accordingly.

However, the plurality of worlds expresses only one aspect of agents' relative ignorance. As we have said, we also assume to work with a plurality of accessibility relations. The resulting semantics is as follows:[11]

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[9] *Ibidem*.

[10] R. Fagin et al., *Reasoning..., op. cit.*; J.J. Meyer, *Modal Epistemic..., op. cit.*

[11] Semantics for non-normal modal logics have a long and distinguished tradition which goes back to the work of D. Scott, *Advice in Modal Logic*, [in:] R. Hilpinen (ed.), *Philosophical Problems in Logic*, Reidel, Dordrecht 1970, p. 143-173; R. Montague, *Universal Grammar*, "Theoria" 1970, vol. 36, p. 373-398; K. Segerberg, *An Essay*

Definition 1 (Multi-relational semantics: frames, models and meaning of $\Box$).[12] Multi-relational semantics is based on the following notions:

- A *multi-relational frame* is a tuple $\mathcal{F} := \langle W, \mathcal{R} \rangle$ where W is a non-empty set and $\mathcal{R}$ is a (possibly infinite) set of binary relations on W.
- A *multi-relational model* is a tuple $\mathcal{M} := \langle W, \mathcal{R}, V \rangle$ where $\langle W, \mathcal{R} \rangle$ is a multi-relational frame and V is a function (assignment) $V : Prop \rightarrow \mathcal{P}(W)$.
- Given a multi-relational model $\mathcal{M} := \langle W, \mathcal{R}, V \rangle$, a propositional letter p is true under V in a state $w \in W$ iff $w \in V(p)$. The truth conditions for all boolean operations are standard. The condition to evaluate $\Box$-formulae is as follows. For any $w \in W$:
  $w \vDash v \Box A$ iff $\exists R_i \in \mathcal{R} : \forall x (w R_i x \Leftrightarrow x \vDash_V A)$.
- Given a multi-relational frame $\mathcal{F} := \langle W, \mathcal{R} \rangle$, a model $\mathcal{M} := \langle \mathcal{F}, V \rangle$ and a formula A, we say that $\mathcal{M}$ *satisfies* A iff for some world w, $w \vDash_V A$; A is *true* in $\mathcal{M}$, $\mathcal{M} \vDash_V A$, iff for all w, $w \vDash_V A$; A is *valid* on $\mathcal{F}$, $\mathcal{F} \vDash A$, iff for any model $\mathcal{M}$ on $\mathcal{F}$, $\mathcal{M} \vDash A$. Given a class of frames $\mathbb{F}$, A is $\mathbb{F}$-*valid*, $\mathbb{F} \vDash A$, iff for any frame $\mathcal{F} \in \mathbb{F}$, $\mathcal{F} \vDash A$.
- *Abbreviations.* For any formula A,
  Truth:[13] $\| A \|_V := \{ w \mid w \vDash_V A \}$
  Validity: $\| A \| := \{ w \mid w \vDash_V A$, for any valuation $V \}$.

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in Classical Modal Logic, Filosofiska Studier, Uppsala 1971. Such modal logics are usually placed within the so-called neighborhood semantics, which is slightly different from the multi-relational one of Definition 3.1. Neighborhood semantics considers a set of collections of worlds related to w instead of connecting worlds via an accessibility relation. These collections are the *neighborhoods* of w. Formally, a frame is a pair $\langle W, N \rangle$ where W is a set of possible worlds and N is a function assigning to each w in W a set of subsets of W (the neighborhoods of w). A model is thus a triple $\langle W, N, V \rangle$, where $\langle W, N \rangle$ is a frame and V is a valuation function: $\Box A$ is true at w iff the set of elements of W where A is true is one of the sets in $N(w)$; i.e. iff it is a neighborhood of w. Despite the mathematical perspicuity of this semantics, neighborhood models are often considered far from intuitive when applied to provide philosophical insights of epistemic logics.

[12] G. Governatori and A. Rotolo, *On the Axiomatization of Elgesem's Logic of Agency and Ability*, "Journal of Philosophical Logic" 2005, vol. 34 (4), p. 403-431.

[13] When clear from the context, we will omit the reference to V.

Multi-relational semantics was originally proposed by Schotch and Jennings[14] and Goble[15]. If we just confine our attention to multi-relational modal frames, the question is how to philosophically interpret such a plurality of relations in epistemic logics. (Semantic structures are based on possibly infinite sets of accessibility relations.) In other words: what can we say about the intuitive reading of such multiplicity of criteria for selecting epistemic alternatives? Originally, multi-relational semantics was developed in the field of deontic logic. In deontic logic, Kripke's accessibility relation selects for each world those states of affairs that are (morally, legally, etc.) ideal with respect to it: hence, if $\Box A$ is true in a world w, this simply means that A is the case in all ideal alternatives to w. The interpretation of multi-relational models, as given for example in deontic logics, is thus that each accessibility relation corresponds to a particular 'standard of value' or a norm that selects those ideal worlds; however, it is not guaranteed that such worlds are still ideal according to different standards of value or norms, namely, according to different accessibility relations. From this perspective, different relations correspond to different deontic standards or that conflicting norms are obtained from otherwise consistent different systems of norms.

If we import this intuition in the domain of epistemic logics, the multiplicity of relations may express the idea that there exist many epistemic standards and that the truth conditions for knowledge assertions can vary across contexts as a result of shifting epistemic standards. The idea of plurality of epistemic standards[16] was defended within different philosophical theories of knowledge,[17] none

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[14] P.K. Schotch and R.E. Jennings, *Non-Kripkean Deontic Logic*, [in:] R. Hilpinen (ed.), *New Studies in Deontic Logic*, Reidel 1981, p. 149-162.

[15] L. Goble, *Multiplex Semantics for Deontic Logic*, "Nordic Journal of Philosophical Logic" 2001, vol. 5 (2), p. 113-134; and by the same author: L. Goble, *Preference Semantics for Deontic Logic*, Part II: *Multiplex Models*, "Logique et Analyse" 2004, vol. 47, p. 113-134.

[16] J.L. Pollock, *Contemporary Theories of Knowledge*, Rowman & Littlefield, Savage 1986, p. 190-193.

[17] N. Malcolm, *Knowledge and Belief*, "Mind" 1952, vol. 61, p. 178-189; A. Goldman, *Discrimination and Perceptual Knowledge*, "The Journal of Philosophy" 1976, vol. 73,

of which should be necessarily assumed to confer a minimal philosophical meaning to epistemic multi-relational models. Let us just consider how Hector-Neri Castañeda[18] illustrates what a plurality of epistemic standards means and how it may affect the truth conditions of knowledge assertions:

Example 1 (Discovering America example adapted after H. Castañeda[19]). 'What counts as knowing' that Christopher Columbus discovered America on October 12, 1492 might change depending on whether we are considering (i) a television quiz show, (ii) a high school student's essay, or (iii) a defence of the traditional dates of America's culture from some famous Harvard historian. Hence, we have in this example three epistemic standards. The fact that

$$\Box \text{ (Columbus discovered America on October 12, 1492)} \qquad (1)$$

is true according to, for example, standard (i) does not entail that it is also true according to standard (iii), which is somehow more demanding. Hence, in general, we could tolerate epistemic expressions such as:

$$\Box A \wedge \Box \neg A \qquad (2)$$

because different standards can lead to know that Columbus discovered America or to know that this was plainly false.

Indeed, if we do not impose any special condition on multi-relational frames, then we have the modal system E. In this setting, it is easy to check that formula (2) is not contradictory (see Figure 2 below). A simple inspection of the figure shows that this is possible because an agent can know/believe that A is true and know/believe

p. 771-791; R. Rorty, *Philosophy and the Mirror of Nature*, Princeton University Press, Princeton 1979.

[18] H. Castañeda, *The Theory of Questions, Epistemic Powers, and the Indexical Theory of Knowledge*, "Midwest Studies in Philosophy" 1980, vol. 5, p. 217.

[19] *Ibidem*.

that it is false because the worlds selected by R_1 (the epistemic alternative v selected by the standard R_1) make A true, while the worlds selected by R_2 (the epistemic alternative z selected by the standard R_1) make A false.

Hence, if we interpret relations as different epistemic standards, it is not required that the truth of (2) corresponds to a genuine *cognitive dissonance*,[20] because there is no real epistemic conflict between $\Box A$ and $\Box\neg A$: each formula refers to a different standard. A true cognitive dissonance occurs rather when $\Box(A \wedge \neg A)$ is true, because this sentence means that there is a logical conflict within a same standard.

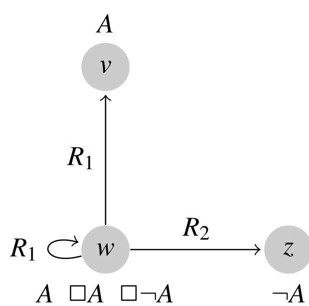

Figure 2: A simple model validating (2)

Let us assume now to formalize Example 1 above following the above semantic intuitions.

Example 2 (Discovering America, cont'd). Let us denote 'Columbus discovered America on October 12, 1492' with A and represent standards as follows:

(i)	a television quiz show	=	R_1
(ii)	a high school student's essay	=	R_2
(iii)	a defence of the traditional dates of America's culture from some famous Harvard historian	=	R_3

[20] E. Aronson, *The Theory of Cognitive Dissonance: A Current Perspective,* [in:] L. Berkowitz (ed.), *Advances in Experimental Social Psychology*, vol. 4, Academic Press, New York 1969.

For formula

$$\Box A \tag{3}$$

it is sufficient that A is true in all worlds selected by one standard, as the model in Figure 3 shows.

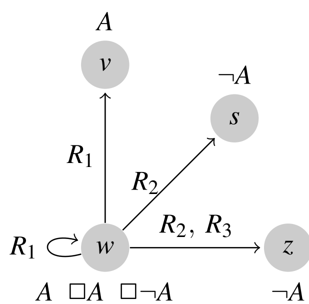

Figure 3: A simple model illustrating Castañeda's example

This analysis suggests that the agents' ignorance is not only captured by having more alternatives for a given world, but also by having more standards. In fact, the standards (i), (ii), and (iii) of Example 1 and 2 represent different contexts as well as 'perspectives' of knowledge, which overall express the fact of a structural bounded epistemic capability with regard to the time when America was discovered. For an omniscient agent g, it would be odd to argue that from a certain perspective g knows that A is false, while from another perspective he knows that A is true, because an omniscient being is supposed to know precisely what is objectively true; hence, a multiplicity of epistemic standards reflects a certain degree of ignorance, at least insofar as the absence of ignorance is taken to correspond to omniscience. Indeed, reducing the number of relations lessens the structural degree of ignorance of agents and leads to a higher degree of agents' omniscience.

3.2. A Stairway to omniscience: an easy step after E

As outlined in Figure 1, the first step in the path that leads from E to full divine logical omniscience is adding schema **M**, i.e.[21]

$$\mathbf{M} := \Box(A \wedge B) \rightarrow (\Box A \wedge \Box B).$$

This schema seems relatively acceptable in epistemic logic. First of all, its validity is assumed in most non-normal modal systems – it is actually discarded only by the system E. Second, the schema looks conceptually harmless: if I know/believe both sentences together, at the same time, then it must be also true that I know/believe that America was discovered by Columbus on October 12, 1492 and that I know/believe that Betsy Ross reported in May of 1776 that she sewed the first American flag. Semantically, the following result holds for **M**:[22]

Proposition 1 (schema M). *For any multi-relational frame $\mathcal{F}$, and any world w and relation R_i, let us $R_i(w)$ denote the set of worlds accessible from w via the relation R_i. Let J and K be subsets of W in $\mathcal{F}$. Hence, the following holds:*
$\mathcal{F} \vDash \Box(A \wedge B) \rightarrow \Box A \wedge \Box B$ iff $\mathcal{F}$ is supplemented, i.e. for any valuation V, for any world $w \in W$, if there exists a relation R_i such that $R_i(w) = J \cap K$, then there are two relations R_j and R_k such that $R_j(w) = J$ and $\mathrm{R_k}(w) = K$.

In other words, if there is one epistemic standard according to which *A* and *B* are jointly true, there are two standards that validate respectively *A* and *B*.

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[21] A different but mathematically equivalent route can be taken by discussing stronger inference rules than **RE**, starting with **RM**. However, referring axiom schemata is much more philosophically perspicuous to discuss the notion of divine omniscience. Hence, we will mostly follow this second route.

[22] G. Governatori and A. Rotolo, *On the Axiomatization..., op. cit.*

Consider the following example:

Example 3 (Columbus and Betsy Ross). Let us denote 'Columbus discovered America on October 12, 1492' with P and 'Ross reported in May of 1776 that she sewed the first American flag' with Q. Again, let us suppose to work with the mentioned epistemic standards:

(i)	a television quiz show	=	R_1
(ii)	a high school student's essay	=	R_2
(iii)	a defence of the traditional dates of America's culture from some famous Harvard historian	=	R_3

For formula

$$\Box(P \wedge Q) \rightarrow (\Box P \wedge \Box Q) \tag{4}$$

it is sufficient to have supplemented models (see Proposition 1) such as in Figure 4.

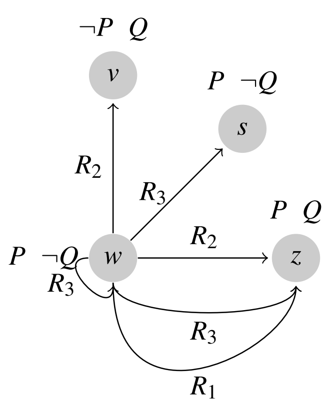

Figure 4: A simple model $\mathscr{M}$ illustrating Example 3

In the model $\mathscr{M}$ represented in Figure 4, for any $x \in W$, we have that x $\vDash \Box(P \wedge Q)$ because there is a relation R_1 such that $R_1(w) = \{z\} = \| P \| \cap \|Q \|$. All epistemic alternatives that make both P and

Q true are related to w via the perspective of the standard (i) a television quiz show; hence, it is true in w that the given agent knows/believes that $P \wedge Q$ is the case. Also, the other two standards, (ii) a high school student's essay and (iii) a defence of the traditional dates of America's culture from some famous Harvard historian connect w respectively to precisely those worlds that make true the sentences P (via R_3) and Q (via R_2), hence:

- $\Box(P \wedge Q)$ is true in w (via standard (i), i.e. relation R_1), $\Box P$ is true in w (via standard (iii), i.e. relation R_3) and $\Box Q$ is true in w (via standard (ii), i.e. relation R_2)
- for any other world $x \in \{v, s, z\}$, we have that $\nvDash \Box(P \wedge Q)$; therefore
- formula (4) is true in $\mathscr{M}$.

3.3. More and harder steps: conflicts, coherence and epistemic paradigms

There is a further important schema that plays a central role in our quest for a logical definition of the concept of omniscience from the perspective of epistemic systems, namely, the schema **C**:

$$\mathbf{C} := (\Box A \wedge \Box B) \rightarrow \Box(A \wedge B).$$

Adding **C** to the formerly defined system M generates the system R, the smallest regular modal logic. This system shows very interesting properties. Let us focus on **C**. If there are two standards guaranteeing, respectively, that $\Box P$ and $\Box Q$ are true, then there is possibly a third standard that selects all the epistemic alternatives in which $P \wedge Q$ is true, namely $\Box(P \wedge Q)$ holds. In general, the result for **C** is the following:

Proposition 2 (schema C). *For any multi-relational frame $\mathscr{F}$, assuming the following holds:*

$\mathcal{F} \models \Box A \wedge \Box B \rightarrow \Box(A \wedge B)$ iff $\mathcal{F}$ is closed under intersections, i.e. for any valuation V, for any world $w \in W$, if there are two relations R_j and R_k such that $R_j(w) = J$ and $R_k(w) = K$, then there exists a relation R_i such that $R_i(w) = J \cap K$.

Example 4 (Columbus and Betsy Ross, cont'd). For formula

$$(\Box P \wedge \Box Q) \rightarrow \Box(P \wedge Q) \tag{5}$$

it is sufficient to have structures closed under intersections (see Proposition 2). Notice that the model in Figure 4 also validates (5). However, consider a subtle variation, as depicted in Figure 5.

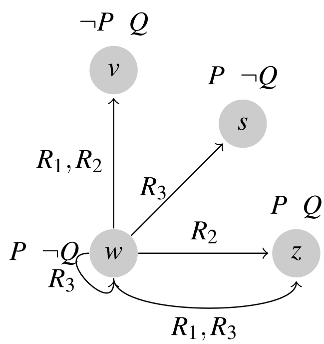

Figure 5: A variation $\mathcal{M}$' of the model $\mathcal{M}$ of Figure 4 that validates (4) but falsifies (5)

The model $\mathcal{M}$' in Figure 5 still validates (4). However,

- $\Box P$ is true in w (via standard (iii), i.e. relation R_3) and $\Box Q$ is true in w (via standard (ii), i.e. relation R_2)
- $\Box(P \wedge Q)$ is false in w because $\| P \| \cap \| Q \| = \{z\}$ but there is no accessibility relation R_j such that $R_j(w) = \{z\}$.
- formula (5) is false in w and not valid in $\mathcal{M}$'.

Here, we may have indeed two epistemic standards that individually support the agent's knowledge/belief that 'Columbus discovered America on October 12, 1492' and 'Ross reported in May of 1776 that she sewed the first American flag' are true, but it is far from obvious that there is a standard that supports them jointly. On the other hand, the difficulty in saying that there is such a standard for $P \wedge Q$ does not undermine the truth of (4), since, if there is no such relation, then the formula is trivially true in w (its antecedent is false).

Notice that schema **C** plays a crucial role in enforcing cognitive dissonances and in making explicit epistemic conflicts. Indeed, let us take Example 4 and replace Q with $\neg P$. Hence, we can simply consider the following instance of **C**:

$$(\Box P \wedge \Box \neg P) \rightarrow \Box(P \wedge \neg P) \tag{6}$$

Since (6) is true for example in w, then there is at least an epistemic standard (represented in the example and in Figure 4 by R_1) that connects w to all epistemic alternatives that make $P \wedge \neg P$ true. However, $P \wedge \neg P \equiv \bot$, hence the standard refers to a contradiction, which makes void R_1 and hence we should have that $R_1(w) = \emptyset$. In a different but related perspective, since the modal system R makes valid the inference rule **RR**, i.e. $\vdash A \wedge B \rightarrow C \Rightarrow \vdash \Box A \wedge \Box B \rightarrow \Box C$[23], then, if we have $\Box P$ and $\Box \neg P$, we obtain $\Box X$ for any sentence X: $(P \wedge \neg P) \rightarrow X$ is in fact a tautology of CPC. Hence, suppose we know/believe that P and know/believe that $\neg P$. We could obtain $\Box Q$, $\Box \neg Q$, $\Box(Q \wedge \neg Q)$, $\Box$(*Bologna is in tke UK*), and so forth.

4. A different path: truth and logical omniscience

We have discussed in Section 3 some very weak epistemic logics. In Section 3.3, however, we highlighted that combining schemata **M** and **C** results in the well-known modal system R, where a much stronger version of logical omniscience emerges: here, we can easily include any tautology and logical truth in God's knowledge base as well as making explicit any cognitive dissonance.

[23] B.F. Chellas, *Modal Logic*, *op. cit.*, Chapter 2.

A different (and not equivalent) path can be taken to capture divine understanding by assuming **M** and state that God knows the Truth. This last statement is expressed by the axiom schema known in the alethic tradition as the Necessity of Truth:

$$\mathbf{N} := \Box\top.$$

As formerly observed, the propositional constant $\top$ is taken to mean the Truth and its truth value is, accordingly, always true. Notice that the schema **N** is enough to include any tautology and logical truth in God's knowledge base. Knowing only one theorem, only one logical Truth would be sufficient to know all the classical theorems. Indeed, for any theorem A it holds that $A \equiv \top$ and it is enough to apply **RE** and **MP** to derive $\Box A$. Hence, it is sufficient to add the schema **N** to the system E to state that God knows all the truths of logic, i.e. all the theorems generated within the system. This intuition is usually captured by the rule **RN**:

$$\mathbf{RN} := \mathrm{A} / \Box A$$

This rule is derivable in the system NM, the smallest N-monotonic system, obtained by adding **N** to M:

Lemma 1. *The rule* **RN** *is derivable in the system* NM.

Proof.

$\vdash_{\mathsf{NM}} A$	Assumption
$\vdash_{\mathsf{NM}} A \equiv \top$	
$\vdash_{\mathsf{NM}} \Box A \equiv \Box\top$	**RE**
$\vdash_{\mathsf{NM}} \Box\top$	Axiom **N**
$\vdash_{\mathsf{NM}} \Box A$	**MP**

The most interesting feature about NM-systems is that it enables the switch from multi-relational strong semantics to weak frames. In fact, the logic NM is known to be sound and complete with respect to weak multi-relational frames, i.e. structures in which the truth condition for the $\Box$ operator is evaluated as follows:

Definition 2 (Weak truth conditions for multi-relational frames). For any $w \in W$: $w \vDash \Box A$ iff $\exists R_i \forall x(wR_i x \Rightarrow x \vDash A)$.

The validity of the schemata **M** and **N** follows immediately by the adoption of this evaluation clause. Notice that the schema **N** states something rather strong. It says, in fact, the any agent operating within the system knows all the truths of logic, all the theorems. However, this type of omniscience still concerns the abstract truths of mathematics rather than contingent facts. This difference becomes rather more evident if looked at from a semantic perspective. What the schema claims, in fact, is that an agent knows all *valid* propositions, i.e. those formulae which are true *everywhere*, in all possible worlds of all possible frames and under all possible valuations. On the other hand, if a fact happens to be true in a specific state of a model, under a specific valuation (but it can still be false under other conditions), there is no way-yet-to infer that an agent knows it.

5. Normal modal logics and the nature of knowledge and knowers

So far we have presented a semantic scenario designed to accommodate different epistemic perspectives and paradigms. However, this looks like a feature suited to depict the sort of knowledge possessed by humankind rather than divinity. In fact, given the laws of CPC, we are bound to accept that any proposition has one and, even more importantly, only one truth value: the law of excluded middle $A \vee \neg A$ is a classical tautology. Semantically, this is mirrored by the fact that the intersection of two complementary sets of epistemic alternatives is always empty. Hence, no genuine epistemic standard (i.e. a relation which is not empty) can accommodate both A and $\neg A$. That known facts should be coherent is suggested, as we have already said, by the schema **C**. It states is that if an agent knows two distinct facts and such facts are contradictory, then he must also use a further epistemic standard which is trivial, i.e. a standard which makes him believe everything (semantically: an empty binary relation). On the other hand, if the two facts are indeed consistent with each other (semantically: the intersection of $\| A \|$ and $\| B \|$ is not empty), then, by **C**, the agent must possess another epistemic standard to accommodate both propositions. In general, for any couple of genuine epistemic paradigms, there must exist a third one which takes into account those facts that are common to both. This means that *true* knowledge is consistent and cannot handle contradictions, i.e. all non-trivial epistemic paradigms are coherent with each other. Intuition would suggest that this is equivalent to possessing only one epistemic standard and this is perfectly consistent with our idea of divine knowledge. Semantically, a multi-relational weak frame with only one binary relation is called a Kripke frame.

Definition 3 (Kripke semantics). A Kripke frame is a multi-relational frame $\mathscr{F} := \langle W,\mathscr{R}\rangle$, such that the cardinality of $\mathscr{R}$ is 1.
Given a model $\mathscr{M} := \langle W,\mathscr{R},V\rangle$, a world w and a wff A, $w \vDash_V \Box A$ iff for any v, if wRv then $v \vDash_V A$.

The theory generated by adding **RE**, **M**, **N**, **C** to CPC is precisely K, a system which is known to be sound and complete with respect to Kripke frames.

Modal logics above K are called *normal*. It is often argued that from the epistemic perspective normal systems are too strong to model human agents: despite the manageability of such logics for AI applications and their low computational complexity, normal epistemic logics raise a number of difficulties if employed to philosophically clarify the nature of human knowledge and belief. One of the most well-known problems is that normal epistemic logics are affected by various forms of logical omniscience, which looks mostly unsuitable for modeling human epistemic capabilities.[24] Nevertheless, if one's goal is to model divine omniscience, normal logics seem no longer too strong, but rather not strong enough. In the following section, we shall further investigate this issue, by considering further axiom schemata within a normal-modal-logic setting.

5.1. Some well-known schemata

Three well-known axiom schemata are widely discussed in the context of epistemic logics:[25] they all contribute to characterising the concept of omniscience and can describe aspects of divine knowledge as well.

5.1.1. Knowledge vs belief: God's infallibility

A first schema is the following:

$$\mathbf{T} := \Box A \to A,$$

which can be called the Principle of Truthful Knowledge. It claims that an agent cannot be mistaken when he knows something: what is known is also true. The contrapositive of this schema is $\mathbf{T}^* := A \to \neg\Box\neg A$ and states that given a true fact, it is impossible to know the opposite. This schema is traditionally considered as defining knowledge versus belief: if the logic for $\Box$ does not contain **T**, then $\Box$ represents agents' belief but not their knowledge.[26] Indeed, it would be

[24] R.Fagin et al., *Reasoning...*, *op. cit.*; J.J. Meyer, *Modal Epistemic...*, *op. cit.*

[25] R. Fagin et al., *Reasoning...*, *op. cit.*; J.J. Meyer, *Modal Epistemic...*, *op. cit.*

[26] J. Hintikka, *Knowledge and Belief: An Introduction to the Logic of the Two Notions,* Cornell University Press, Ithaca–New York 1962.

quite odd to assume that anything believed is also true, whether the same is acceptable, if not desirable when modelling knowledge.

If applied to modelling God's omniscience, **T** has the following immediate positive effects:

- If all beliefs of God are expressed with $\Box$, then we capture the idea that God is epistemically infallible: none of God's beliefs can, in fact, be false when **T** is accepted. Although this schema is not directly linked to the idea of omniscience, it plays anyway an indirect but important role in characterising God's knowledge.[27]
- The fact that God is epistemically infallible alleviates the problem described by the end of Section 3.3: if we have $\Box P$ and $\Box\neg P$, then we can derive $\Box X$ for conclusions X that are factually false. Schema **T** guarantees that whatever X is known/believed by the agent is also true.
- Since modelling God's epistemic nature cannot admit the distinction between knowledge and belief, otherwise we should admit that God believes things that are not true, and so He is not infallible; then, if $\Box$(*Bologna is in tke UK*), then Bologna is indeed in the UK.
- If this last conclusion, i.e. 'Bologna is in the UK', looks quite odd when we model human or non-divine agents, it should be seen as positive with divine knowledge: the mere fact that God knows that any X is true implies (or, perhaps, makes) such an X true.

This analysis is, hence, in line with the idea that God believes no falsehoods, which was pointed out, for instance, by Plantinga[28] and Gale[29].

[27] P. Weingartner, *Omniscience: From a Logical Point of View*, Ontos Verlag, Frankfurt–Paris–Ebikon–Lancaster–New Brunswick 2008, Chapter 1.
[28] A. Plantinga, *God, Freedom, and Evil*, "Eerdmans" 1977, vol. 20.
[29] R. Gale, *On the Existence and Nature of God*, Cambridge University Press, Cambridge 1991.

5.1.2. God's knowledge about His own knowledge (and ignorance)

The other two schemata are the following:

4 := $\Box A \rightarrow \Box\Box A$
5 := $\neg\Box A \rightarrow \Box\neg\Box A$

The epistemic common interpretation of **4** is that it expresses the Principle of Positive Introspection. This states that an agent that knows (or believes, if **T** does not hold) something, is also aware of this fact. This form of positive awareness should apply also to God's knowledge:[30] if what God knows is true, then, if it is true that He knows some *A*, He should also know that He knows that.

Schema **5** expresses the Principle of Negative Introspection. This states that an agent who ignores a fact is also aware of his ignorance. Of course, this axiom captures a form of omniscience as well, even though it could sound strange that God may not know that some *A* is the case, because this would mean that God does not know *A*. Indeed, this schema does not force God to be necessarily ignorant, but states that, *if* God were ignorant with respect to some *A*, *then* he would be aware of this fact. However, the schema does not logically exclude, too, that God could be ignorant with respect to any true *A*. This is not guaranteed by any system containing only **T**, **4**, and **5**. We will discuss this question in the subsequent sections.

5.2. The S5 system: the heaven of knowledge?

The system resulting by adding **T**, **4** and **5** to the minimal normal logic K is usually called S5 and it has been widely studied throughout the years. Some modal schemata correspond to specific properties of

[30] P. Weingartner, *Omniscience...*, *op. cit.*, Chapter 12.

frames. For instance, we have that **T** characterises the class of reflexive frames, **4** isolates that of transitive ones and **5** that of Euclidean structures. The schema **B** $:= A \rightarrow \Box\Diamond A$, which is derivable in S5, characterises the class of symmetric frames. These observations lead to the well-known result stating that S5 is sound and complete with respect to the class of Kripke structures whose binary relation is an equivalence, i.e. R is:

- Reflexive: $\forall w(wRw)$
- Symmetric: $\forall w \forall v(wRw \Rightarrow vRw)$
- Transitive: $\forall w \forall v \forall z(wRv \ \& \ vRz \Rightarrow wRz)$.

An interesting and well-known technical result states that the system S5 is also determined by the class of frames whose relation is *universal*[31]. This means simply that all worlds are connected to each other, and hence the accessibility relation no longer exists. Universal frames can be seen as epistemic islands, monads in which knowledge does not depend on the perspective of agents. Moreover, from the agents' point of view, a plurality of epistemic alternatives is no longer a theoretic possibility, since all worlds are epistemically equivalent, thus making the notion of epistemic alternative void.

Within a universal frame, only one epistemic paradigm is accepted and the information must be consistent. Semantically, this is mirrored by the fact that the axiom **T** imposes reflexivity, hence a type of seriality: this guarantees that agents can never have contradictory knowledge, i.e. the situation $\Box A \rightarrow \Box\neg A$ is no longer acceptable. S5 frames cannot handle conflicts, nor cognitive relativism, since we have that $\Box A \rightarrow \neg\Box\neg A$ holds, too. Hence, these are structures designed to accommodate perfect, logically omniscient epistemic agents.

Nevertheless, although S5 expresses the strongest version of logical omniscience, it is still too weak to capture God's omniscience.

31 B.F. Chellas, *Modal Logic*, *op. cit.*, p. 97-98.

The heaven of knowledge is not far away, but we still have to make a substantial step beyond normal logics in order to climb to the top of the stairway to divine knowledge.

6. Factual omniscience and divine knowledge

So far, we have characterised the S5 epistemic agent as perfectly analytical, aware of all the consequences of his knowledge, logically omniscient, never mistaken, consistent, perfectly conscious of what he knows and ignores. But is it enough to characterise an omniscient being?

Another type of omniscience, we have said, can be mentioned: the so-called factual omniscience. This is often characterised as maximal or complete knowledge. Formally speaking, a set of sentences is said to be maximal if, for any well-formed formula A, either A or its negation $\neg A$ belongs to the set itself. Thus an *epistemically maximal* knowledge base should be such that, given a formula A, it contains either A or $\neg A$. Hence, an agent possessing maximal knowledge knows any *true* fact, i.e. for any proposition A either $\Box A$ or $\Box\neg A$ belongs to the theory. CPC assumes that any proposition has a semantic univocal truth value, either true or false (mirrored syntactically by the law of the excluded middle). That an omniscient being may lack the knowledge of any true (respectively, false) fact seems to contradict the concept of omniscience itself. Therefore, it sounds not only reasonable, but actually necessary to add a further schema to our epistemic theory S5 to capture this intuition. Formally, such an axiom should look like an epistemic version of the law of the excluded middle, namely:

$$\mathbf{FK} := \Box A \vee \Box\neg A,$$

the acronym **FK** standing for *factual knowledge*, as it seems to formalise *factual omniscience*, i.e. the knowledge of any *true* fact. By adding **FK** to S5, the resulting system should be sufficiently strong to give a good account of God's omniscience. Indeed, the idea of divine omniscience as expressed by **FK** was informally discussed, for instance, by Zagzebski.[32]

The trouble is that, upon a closer analysis, it may actually appear rather too strong. Indeed, a very troublesome schema is easily derivable by means of a rather trivial logical deduction:

$\vdash_{S5.FK} \Box\neg A \vee \Box A$	**FK**
$\vdash_{S5.FK} \neg\Box\neg A \rightarrow \Box A$	Tautology
$\vdash_{S5.FK} A \rightarrow \neg\Box\neg A$	Axiom **T***
$\vdash_{S5.FK} A \rightarrow \Box A$	Law of concatenation

The conclusion $A \rightarrow \Box A$ is known as Anselm's Principle (**AP**) as it is used in the modal version of the famous ontological argument expressed by Anselm.[33] Alethically, it states that anything which happens to be true is also necessary; epistemically, that any true fact is known. A reader not familiar with modal logic may not notice the profound difference between the Anselm's Principle and the rule **RN** := $A/\Box A$. However, these schemata state indeed two very different things. The **RN** rule is only concerned with logical truth and theorems: it states that whenever a formula happens to be a theorem or, semantically, *valid*, i.e. true under any assignment, then it is also known. Anselm's Principle, on the other hand, states that any formula which is possibly true under some assignment is also known. This captures the inner difference between factual and logical omniscience: any God-like omniscient being should certainly possess both.

The central point is, however, that along with the schema **T** := $\Box A \rightarrow A$, Anselm's Principle brings the system to be trivial as there is a complete collapse of modality: $\Box A \equiv A$. In this scenario, modalities turn out to be empty and the system is actually equivalent to CPC.

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[32] L. Zagzebski, *Omniscience, op. cit.*, p. 262.

[33] J. Ernst, *Charles Hartshorne and the Ontological Argument*, "Aporia" 2008, vol. 18 (1), p. 57-66.

Hence, the trouble with **FK** (and with **AP**) is that it rather appears too strong, since the modal concept of knowledge vanishes.

The system obtained by adding **AP** and **T** or, equivalently, **D** and $A \equiv \Box A$ to K is usually called Triv[34] and it is equivalent to CPC. Moreover, the schemata **AP** and **FK** we discussed above are perfectly equivalent under K plus **T**.

However, these schemata express rather different properties. Where **AP** states that whatever is the case, it is also known, schema **FK** says something stronger: given any possible fact, the agent knows either the fact itself or its negation. Their difference is mirrored semantically.

Lemma 2 (Characterisation of AP in Kripke semantics). *For any Kripke frame* $\mathscr{F} := \langle W, \mathscr{R} \rangle$, $\mathscr{F} \vDash A \rightarrow \Box A$ *if and only if for any world* $w \in W$, if $R(w) \neq \emptyset$, *then* $R(w) = \{w\}$.

Lemma 3 (Characterisation of FK in Kripke semantics). *For any Kripke frame* $\mathscr{F} := \langle W, \mathscr{R} \rangle$, $\mathscr{F} \vDash \Box A \vee \Box \neg A$ *if and only if for any world* $w \in W$, $R(w) \neq \emptyset$, *then* $|R(w)| \leq 1$.

These conditions show that **FK** entails the validity of **AP**, whereas the converse does not hold, unless the frame is reflexive (and hence validates **T**). Therefore, it is not true, in general, that the two principles are logically equivalent, as maintained by Wierenga:[35] this happens only if we assume the omniscient being to be epistemically infallible.

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[34] G. Hughes and J. Cresswell, *A New Introduction to Modal Logic*, Routledge, London–New York 1996, p. 64-65.

[35] E. Wierenga, *Omniscience*, [in:] E.N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy*, Spring 2010.

7. Summary

In this paper, we have analysed the concept of divine omniscience within the setting of propositional epistemic logics. We considered different aspects that semantically model the limits for an agent to be fully omniscient:

- a plurality of possible worlds or states, a standard idea in epistemic logics that captures the fact that, when an agent does not possess full knowledge in a certain situation w, then we conceptually refer to all the epistemic alternatives to w;
- a plurality of accessibility relations connecting pairs of worlds, which express different epistemic standards; the fact that, for the same agent, we may have that $\Box A$ and $\Box\neg A$ are both true entails that there are at least two standards and that the agent does not possess any true and objective knowledge in regard to A.

This semantic picture was the general framework within which we discussed the epistemic meaning of several axiom schemata and inference rules. The analysis proceeded from the weakest epistemic systems to the strongest ones by adding different schemata step by step. The result depicted an ascending path to divine omniscience that first gave up the assumption that we may have different epistemic standards and, subsequently, removed any accessibility relation connecting worlds. This said, however, we argued that the obtained logic system is still under the threshold of divinity, as it does not yet model the concept of *complete* or *maximal* knowledge. This final concept may be obtained by adopting schemata that, in fact, trivialise the notion of knowledge: epistemic logics collapse into the classical propositional calculus.